<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">Goce Delcev University – Stip, North Macedonia</td></tr>
<tr><td>Topic</td><td colspan="2">Application of Derivatives</td></tr>
<tr><td>Lesson title</td><td colspan="2">Minimizing and Maximizing Problems</td></tr>
<tr><td>Learning objectives</td><td>✓ Students will be able to estimate minimum and maximum values of different sizes using differentiation;
✓ Students will acquire and deal with derivatives of a function;
✓ Students will be able to deal with different problems in everyday life, which require finding minimum or maximum value of a given size;
✓ Students are encouraged to use technology and different software in their work, while considering problem based situations.</td><td rowspan="2">Strategies/Activities
☐Graphic Organizer
☐Think/Pair/Share
☐Modeling
☐Collaborative learning
☐Discussion questions
☐Project based learning
☐Problem based learning

Assessment for learning
☐Observations
☐Conversations
☐Work sample
☐Conference
☐Check list
☐Diagnostics</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>The aim of the lecture is to make students able to calculate derivatives of a function and apply the derivatives to calculate minimum and maximum of given size.

The teacher gives the next problem to the students:

A farmer in his orchard plants has 50 apple trees. Each tree produces approximately 900 apples in a season. The farmer wants to enlarge the orchard and plant more trees, but by his experience, he knows that for each additional tree planted in the orchard, the output per tree drops by 15 apples for each new tree. How many trees the farmer should add to the existing orchard in order to maximize the total output of the trees?</td></tr>
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<table>
<tr><td>Previous knowledge assumed:</td><td>
- polynomials
- polynomial functions
- algebraic equations
- differentiating techniques
</td><td rowspan="2">Assessment as learning
☐Self-assessment
☐Peer-assessment
☐Presentation
☐Graphic Organizer
☐Homework

Assessment of learning
☐Test
☐Quiz
☐Presentation
☐Project
☐Published work</td></tr>
<tr><td>Introduction / Theoretical basics</td><td>
If N is the number of trees and each tree produces approximately m apples in a season, the total outcome P from those trees is P=N*m. Thus, the total outcome that 50 trees give is P=50*900=45 000 apples.

If the farmer plant one more tree, the total outcome will be P=(50+1)*(900-15)=51*885=45 135 apples.

If the farmer plant two more trees, the total outcome will be P=52*(900-2*15)=45 240 apples, etc.

Students can notice that the total outcome is growing up.

Students are encouraged to use Excel and formulas in it for such calculations.

If the farmer plants 10 trees, the outcome is P=60*(900-10*15)=60*750=45 000.

Thus the total outcome is decreasing and although the farmer has more trees then at the beginning, the total outcome is the same as at the beginning.

The excel sheet is the following:
<table>
<tr><th>Number of trees</th><th>Outcome per tree</th><th>Total outcome</th></tr>
<tr><td>50</td><td>900</td><td>45000</td></tr>
<tr><td>51</td><td>885</td><td>45135</td></tr>
<tr><td>52</td><td>870</td><td>45240</td></tr>
<tr><td>53</td><td>855</td><td>45315</td></tr>
</table>
</td></tr>
</table>

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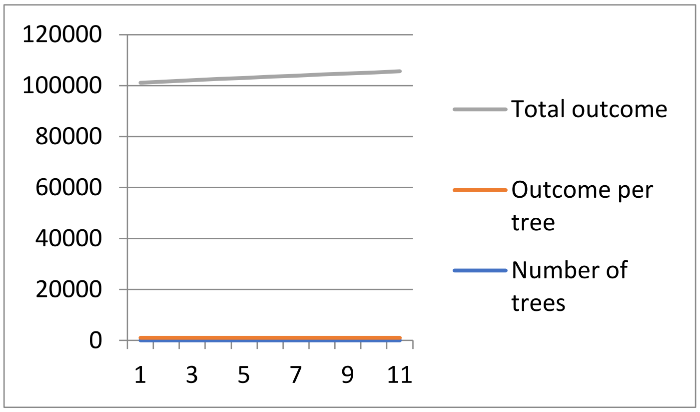
54	840	45360
55	825	45375
56	810	45360
57	795	45315
58	780	45240
59	765	45135
60	750	45000

According to the table, students can realize that the total outcome is greatest when there are 55 apple trees and more trees then 55 contribute in decreasing total outcome.

Students can create different charts with the data in the table, using Excel:



Now, if we give to the students similar problem, but with other numbers, for example 100 trees in the orchard, 1000 apples outcome per tree and 5 apple less for each new tree and encourage them to use again excel for easier calculations (using formulas in it), students will notice that the total outcome is increasing for 10 new trees, for 20 new trees, etc. and for 50 new trees the total outcome is increasing for each new tree. In the spreadsheets they can see that if the farmer plant 51 new trees, the total outcome will be less than the one with 50 new trees, thus it is now starting to decrease.

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Number of trees	Outcome per tree	Total outcome
100	1000	100000
101	995	100495
102	990	100980
103	985	101455
104	980	101920
105	975	102375
106	970	102820
107	965	103255
108	960	103680
109	955	104095
110	950	104500
111	945	104895
112	940	105280
113	935	105655
114	930	106020
115	925	106375
116	920	106720
117	915	107055
118	910	107380
119	905	107695
120	900	108000
121	895	108295
122	890	108580
123	885	108855
124	880	109120
125	875	109375
126	870	109620
127	865	109855
128	860	110080
129	855	110295
130	850	110500
131	845	110695

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132	840	110880
133	835	111055
134	830	111220
135	825	111375
136	820	111520
137	815	111655
138	810	111780
139	805	111895
140	800	112000
141	795	112095
142	790	112180
143	785	112255
144	780	112320
145	775	112375
146	770	112420
147	765	112455
148	760	112480
149	755	112495
150	750	112500
151	745	112495
152	740	112480
153	735	112455
154	730	112420
155	725	112375
156	720	112320
157	715	112255
158	710	112180
159	705	112095
160	700	112000

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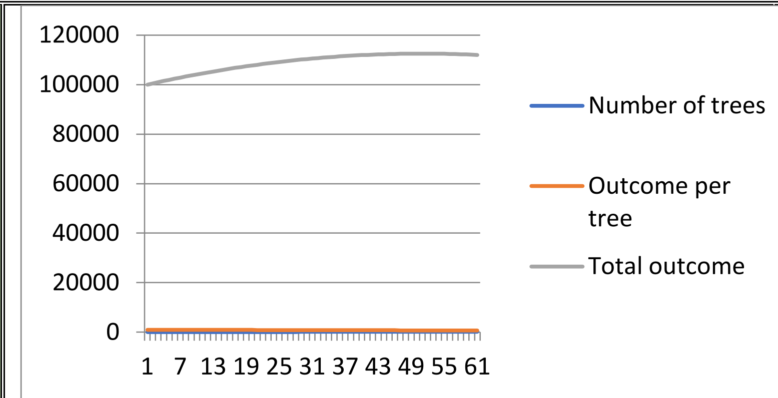

Although the table is easy to be done in Excel and students can easily see the solution, this way is not convenient to solve all similar problems because of the length of the table, which depending of the numbers can reach several pages.

Thus, mathematical method and formulas are necessary for direct calculation of the solution.

Such minimizing / maximizing problems can easily be solved with an application of derivatives of a function.

If $y = f(x)$ is given function with domain D and $x_0 \in D$, let $y_0 = f(x_0)$. If the argument x has been changed for Δx and the new one is $x_0 + \Delta x \in D$, then the value of the function changes to $f(x_0 + \Delta x)$.

Definition: If the limit $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x)}{\Delta x}$ exists, we call it *the first derivative of the function* $y = f(x)$ at the point x_0.

According to the definition, the derivative of a function is related to changes of the values of the argument and the function.
If we use function to represent some size, we can use derivative of a function in a problems related to the changes of the values of that size.

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Using the definition of the first derivative, the derivatives of some elementary functions are calculated and are now used for calculating derivatives of other functions. A table of some elementary functions with their derivatives is the following:

$$f(x)=c, \quad f'(x)=0$$

$$f(x)=x^n, \quad f'(x)=nx^{n-1}$$

$$f(x)=\frac{1}{x}, \quad f'(x)=-\frac{1}{x^2}$$

$$f(x)=\sqrt{x}, \quad f'(x)=\frac{1}{2\sqrt{x}}$$

$$f(x)=e^x, \quad f'(x)=e^x$$

$$f(x)=\ln x, \quad f'(x)=\frac{1}{x}$$

$$f(x)=a^x, \quad f'(x)=ax^{n-1}$$

Let $f(x)$ and $g(x)$ be differentiable functions and *c* is a constant. Then the following equations hold:

$(cf(x))'=cf'(x)$ - constant multiple rule

$(f(x)+g(x))'=f'(x)+g'(x)$ - sum rule

$(f(x)-g(x))'=f'(x)-g'(x)$ - difference rule

$(f(x)\cdot g(x))'=f'(x)\cdot g(x)+f(x)\cdot g'(x)$ - product rule

$\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{(g(x))^2}$ for $g(x)\neq 0$ - quotient rule

Definition: If $f'(x)$ is a first derivative of a function

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	$f(x)$, then its derivative (if it exists), i.e. $\left[f'(x)\right]' = f''(x)$ is called the second derivative of a function $f(x)$. The derivatives can be applied for calculating extreme values of different sizes. If $y = f(x)$ is given function, the function f has its minimum value at if $f'(c) = 0$ and $f''(c) > 0$. The function f has its maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$. If we consider certain size as a function with one variable, we can find its minimum or maximum values with the above rules.	
Action	Let us return to the given problem and construct an appropriate function to find the maximum outcome from the apple trees. Here is the given problem: *A farmer in his orchard plants has 50 apple trees. Each tree produces approximately 900 apples in a season. The farmer wants to enlarge the orchard and plant more trees, but by his experience, he knows that for each additional tree planted in the orchard, the output per tree drops by 15 apples for each new tree. How many trees the farmer should add to the existing orchard in order to maximize the total output of the trees?* Let us denote the total outcome with P and the new trees with x. According to the problem conditions we have: $P(x) = (50 + x)(900 - 15x)$ $P(x) = 45000 - 750x + 900x - 15x^2$ $P(x) = 45000 + 150x - 15x^2$	

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	According to the rules which determine the extreme values, we have to calculate the first derivative and calculate x such that $P'(x)=0$: $$P'(x)=150-30x$$ $$150-30x=0 \quad \Leftrightarrow \quad x=5$$ The second derivative is $P''(x)=-30<0$ thus for $x=5$ the functions reaches its maximum value. So, the farmer has to enlarge his orchard with 5 apple trees in order to maximize the total output. For the other problem which was given to the students, actually the same problem with other numbers, calculating in a same way, students can find: $$P=(100+x)(1000-5x)$$ $$P=100000+500x-5x^2$$ $$P'=500-10x$$ $$P'=0 \quad \Leftrightarrow \quad x=50$$ Thus, in this case the farmer should plant 50 more trees to the existing in order to maximize the outcome.	
Materials / equipment / digital tools / software	Literature given in the references at the end of the document / Digital device which supports Excel / Excel	
Consolidation	With the given examples students can consider that the real functions and their derivatives are important for solving real life problems. Students will learn what is a derivative of a function and how to calculate it. They can learn how to apply differentiation and derivatives to maximize / minimize certain value by given conditions. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.	
Reflections and next steps		
Activities that worked	**Parts to be revisited**	

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Problem solving, collaboration, using technology	Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.
References	

[1] M. L. Bittinger, D. J. Ellenbogen and S.A. Surgent (2012), "Calculus and its applications", Addison-Wesley
[2] G. Strang "Calculus" , Wellelye-Cambridge Press
[3] S. Calaway D. Hoffman and D.Lippman (2014) "Applied Calculus"
[4] *P.D. Lax*, M. S.Terrell (2014) "Calculus with Applications", Springer

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